

# Isomorphic Formulae in Classical Propositional Logic

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## Abstract

Isomorphism between formulae is defined with respect to categories formalizing equality of deductions in classical propositional logic and in the multiplicative fragment of classical linear propositional logic caught by proof nets. This equality is motivated by generality of deductions. Characterizations are given for pairs of isomorphic formulae, which lead to decision procedures for this isomorphism.

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## 1 Introduction

Isomorphism between formulae should be an equivalence relation stronger than mutual implication. This is presumably the relation underlying the relation that holds between propositions that have the same meaning just because of their logical form. Any propositions that are instances, with the same substitution, of isomorphic formulae would have the same meaning, which presumably need not be the case for formulae that are just equivalent, i.e. which just imply each other.

One may try to characterize isomorphic formulae by looking only into the inner structure of formulae. This is the way envisaged by Carnap and Church (see [4], Sections 14-15, where related work by Quine and C.I. Lewis is mentioned, [5], [1], Section 2, and references therein).

Another way is to try to characterize isomorphism between formulae by looking also at the outer structure in which formulae are to be found. This outer structure may be a deductive structure, which is characterized in terms of categories in categorial proof theory. The categories we need are syntactical: their objects are formulae and their arrows are deductions.

Isomorphism between formulae may then be understood exactly as isomorphism between objects is understood in category theory. The formulae  $A$  and  $B$  are isomorphic when there is a deduction  $f$ , i.e. arrow, from  $A$  to  $B$ , and another deduction  $g$  from  $B$  to  $A$ , such that  $f$  composed with  $g$  is equal to the identity deduction from  $A$  to  $A$ , while  $g$  composed with  $f$  is equal to the identity deduction from  $B$  to  $B$ . This analysis of isomorphism presupposes a notion of equality between deductions, which is formalized in our syntactical categories. (*Equality between deductions* stands here for what we and other authors have called elsewhere *identity of proofs*; see [8], [13], Sections 1.3-1.4, and references therein.) Characterizing this notion of equality between deductions is the main task of categorial proof theory, and of general proof theory.

That  $A$  and  $B$  are isomorphic means here intuitively that they function in the same manner in deductions. In a deduction one can replace one by the other, either as premise or as conclusion, so that nothing is lost, nor gained. The replacements, which are made by composing our deduction with the deductions  $f$  and  $g$ , are such that they enable us to return to our original deduction by further composing with  $g$  and  $f$ , since  $f$  composed with  $g$  and  $g$  composed with  $f$  are identity deductions, and hence may be cancelled. (For a view concerning isomorphic formulae like the one presented here, and its relationship with propositional identity, see [6], Section 9, and [7], Section 5.)

The study of isomorphic formulae first started in intuitionistic logic, for which it is widely believed that we have a solid nontrivial notion of equality of deductions. This notion is characterized either in terms of the typed lambda calculus (via the Curry-Howard correspondence), or in terms of categories based on cartesian closed categories (these characterizations may be equivalent; see [19]).

A result exists in this area for the implication-conjunction- $\top$  fragment of intuitionistic logic (see [21]). As far as we know, the latest advances concerning the still open problem of characterizing formulae isomorphic in the whole of intuitionistic propositional logic (which is related to Tarski's high-school algebra problem; see [3]) were made in [2] and [16]. There is a related result characterizing isomorphic formulae in the analogous multiplicative fragment of linear logic, which corresponds to symmetric monoidal closed categories, and is common to classical and intuitionistic linear logic (see [22] and [9]).

The problem of characterizing isomorphic formulae was not approached up to now in classical propositional logic, and the results we are going to present here cover this logic. They cover also a fragment of classical linear propositional logic. To be able to approach our problem, and to obtain significant results, we need for the logics we want to cover a plausible and nontrivial notion of equality of arrows in categories formalizing equality of deductions in these logics. A consensus for classical linear propositional logic may be found around the multiplicative fragment of that logic caught by proof nets, which leads to notions of category closely related to star-autonomous category (see [20], [14] and references therein).

For classical propositional logic, it is on the contrary widely believed that no nontrivial notion of category would do the job. It is believed that no nontrivial notion of Boolean category may be found. This is indeed the case if one wants these Boolean categories to be cartesian closed (see [8], Section 5, [13], Section 14.3, and references therein). But, whereas on the level of theorems classical logic is an extension of intuitionistic logic, it is not clear that this should be so at the level of deductions and of their equality.

If one does not require that Boolean categories be cartesian closed categories, and bases equality of deductions in Boolean categories on coherence results analogous to those available for classical linear propositional logic, a nontrivial notion may arise. The coherence results in question are categorial results analogous to the classical coherence result of Kelly and Mac Lane for symmetric monoidal closed categories (see [17]). They reduce equality of arrows in the syntactical category to equality of arrows in a graphical model category.

Such a nontrivial notion of Boolean category may be found in [13] (Chapter 14), and Section 4 of the present paper deals with isomorphism of formulae engendered by that particular notion. Section 3 of the paper deals with isomorphism of formulae in classical and classical linear propositional logic different from the notion of Section 4. That other notion, which involves graphical model categories quite like those of Kelly and Mac Lane, is motivated by generality of deductions (as suggested by [18]). The notion of Section 4 may also be understood as involving generality up to a point, but the notion of Section 3 does so more consistently. Both notions are however analogous in that they base equality of deductions on equality of arrows in some graphical model category.

The main results of the paper in Sections 3 and 4 are based on some preliminary elementary results concerning classical propositional logic, which are established in Section 2, and occasionally later in the paper. Although these results are not difficult to reach, when they are combined with more advanced results, such as those that may be found in [13], they give a complete characterization of isomorphic formulae in classical and classical linear propositional logic. These characterizations are such that they easily lead to decision procedures for the isomorphisms in question.

This paper is devoted to the problem of characterizing pairs of isomorphic formulae. A related, but different, problem involving isomorphism is to characterize arrows that are isomorphisms, in categories formalizing equality of deductions. (This problem for the conjunction- $\top$  fragment of classical or intuitionistic logic is dealt with in [10].) Although we have not dealt with that second problem explicitly in this paper, a solution for it may easily be inferred from our results.

We will however not dwell on that, in order not to overburden the text with categorial matters. In the whole paper we try to keep the presence of categories to a minimum, and give more prominence to elementary, easily understandable, logical facts. To appreciate the full import of our results the reader should however be acquainted up to a point with certain notions covered in particular

by [13] (and which we cannot possibly expose all here).

## 2 $\wedge\vee$ and $\neg\wedge\vee$ -equivalences

Let  $\mathcal{L}$  be a propositional language generated out of an infinite set of propositional letters, which we call simply *letters*, with the nullary connectives  $\top$  and  $\perp$ , the unary connective  $\neg$ , and the binary connectives  $\wedge$  and  $\vee$ . We use  $p, q, r, \dots$ , sometimes with indices, for letters, and  $A, B, C, \dots$ , sometimes with indices, for the formulae of  $\mathcal{L}$ . Let  $\mathcal{L}_{\wedge, \vee}$ ,  $\mathcal{L}_{\neg, \wedge, \vee}$  and  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  be the propositional languages defined as  $\mathcal{L}$  but with only the connectives in the subscripts.

We envisage also propositional languages extending  $\mathcal{L}$ , in which we have moreover the binary connectives of equivalence  $\leftrightarrow$  and implication  $\rightarrow$ . We may imagine that these two additional connectives are defined in  $\mathcal{L}$ , and we shall not introduce special names for these extended languages.

Let a  $\wedge\vee$ -equivalence be a formula  $A \leftrightarrow B$  where  $A$  and  $B$  are formulae of  $\mathcal{L}_{\wedge, \vee}$ . Consider the formal system  $\mathcal{S}_{\wedge, \vee}$  whose axioms are the  $\wedge\vee$ -equivalences of the following forms:

$$\begin{aligned} & A \leftrightarrow A, \\ & ((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C)), \quad ((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C)), \\ & (A \wedge B) \leftrightarrow (B \wedge A), \quad (A \vee B) \leftrightarrow (B \vee A), \end{aligned}$$

and whose theorems are the  $\wedge\vee$ -equivalences obtained starting from these axioms with the following rules:

$$\begin{array}{c} \frac{A \leftrightarrow B}{B \leftrightarrow A} \qquad \frac{A \leftrightarrow B \quad B \leftrightarrow C}{A \leftrightarrow C} \\ \frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \wedge C) \leftrightarrow (B \wedge D)} \qquad \frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \vee C) \leftrightarrow (B \vee D)} \end{array}$$

A formula is *diversified* when every letter occurs in it not more than once. A  $\wedge\vee$ -equivalence is *diversified* when  $A$  and  $B$  are diversified.

Assume that  $\text{let}A$  is the set of letters occurring in  $A$ , and let  $A_B^p$  be the result of substituting the formula  $B$  for every occurrence of  $p$  in  $A$ . We can prove the following lemmata.

LEMMA 1. *Assume that  $A$  is a diversified formula of  $\mathcal{L}_{\wedge, \vee}$ , that  $B$  is a subformula of  $A$ , and that  $\text{let}A - \text{let}B = \{p_1, \dots, p_n\}$  (where  $\{p_1, \dots, p_n\}$  is empty if  $n = 0$ ). Then there is a sequence  $S_1, \dots, S_n$ , where  $S_i \in \{\top, \perp\}$ , such that  $A_{S_1 \dots S_n}^{p_1 \dots p_n} \leftrightarrow B$  is a tautology.*

PROOF. We proceed by induction on  $n$ . If  $n = 0$ , then  $B$  is  $A$ , and  $A \leftrightarrow B$  is of course a tautology.

If  $n = k+1$ , and  $A$  is of the form  $C \wedge D$  with  $B$  a subformula of  $C$ , then for  $\text{let}C - \text{let}B = \{q_1, \dots, q_m\}$ , where  $\{q_1, \dots, q_m\} \cup \{r_1, \dots, r_l\} = \{p_1, \dots, p_n\}$  for

$m \geq 0$  and  $l \geq 1$ , we have by the induction hypothesis that for some  $S_1, \dots, S_m$  the formula  $C_{S_1 \dots S_m}^{q_1 \dots q_m} \leftrightarrow B$  is a tautology. Hence  $A_{S_1 \dots S_m \top \dots \top}^{q_1 \dots q_m r_1 \dots r_l} \leftrightarrow B$  is a tautology too.

We proceed analogously when  $B$  is a subformula of  $D$ , or when  $A$  is of the form  $C \vee D$ . In the latter case we substitute  $\perp, \dots, \perp$  for  $r_1, \dots, r_l$ .  $\dashv$

LEMMA 2. *If  $B$  is a diversified formula of  $\mathcal{L}_{\wedge, \vee}$  that has a letter  $q$  not in the formula  $A$  of  $\mathcal{L}$ , then  $A \leftrightarrow B$  is not a tautology.*

PROOF. By Lemma 1, for some  $S_1, \dots, S_n$ , where  $S_i \in \{\top, \perp\}$ , we have that  $B_{S_1 \dots S_n}^{p_1 \dots p_n} \leftrightarrow q$  is a tautology. On the other hand,  $A_{S_1 \dots S_n}^{p_1 \dots p_n} \leftrightarrow q$  cannot be a tautology. So  $A_{S_1 \dots S_n}^{p_1 \dots p_n} \leftrightarrow B_{S_1 \dots S_n}^{p_1 \dots p_n}$  is not a tautology, and hence  $A \leftrightarrow B$  is not a tautology.  $\dashv$

For  $A$  a diversified formula of  $\mathcal{L}_{\wedge, \vee}$  we say that the letters  $p$  and  $q$  are *conjunctively joined* in  $A$  when  $A$  has a subformula  $P \wedge Q$  or  $Q \wedge P$  such that  $p$  is in  $P$  and  $q$  is in  $Q$ , and we say that  $p$  and  $q$  are *directly conjunctively joined* in  $A$  when  $A$  has a subformula  $P \wedge Q$  or  $Q \wedge P$  such that  $p$  is in  $P$  and  $q$  is in  $Q$ , no subformula of  $P$  containing  $p$  is a disjunction, and no subformula of  $Q$  containing  $q$  is a disjunction. We define analogously disjunctively and directly disjunctively joined formulae (we just replace  $\wedge$  by  $\vee$  and  $\vee$  by  $\wedge$ ). We can prove the following proposition.

PROPOSITION 1. *A diversified  $\wedge\vee$ -equivalence is a tautology iff it is a theorem of  $\mathcal{S}_{\wedge, \vee}$ .*

PROOF. It is clear that every theorem of  $\mathcal{S}_{\wedge, \vee}$  is a tautology. (This is established by induction on the length of proof in  $\mathcal{S}_{\wedge, \vee}$ .) For the converse, we suppose that the diversified  $\wedge\vee$ -equivalence  $A \leftrightarrow B$  is a tautology, and we proceed by induction on the number  $n$  of connectives in  $A$ . By Lemma 2, this number must also be the number of connectives in  $B$ .

If  $n = 0$ , then, by Lemma 2, the formulae  $A$  and  $B$  must both be a letter  $p$ , and  $p \leftrightarrow p$  is an axiom of  $\mathcal{S}_{\wedge, \vee}$ . If  $n = k + 1$ , then  $A$  has a subformula either of the form  $p \wedge q$  or of the form  $p \vee q$ . Suppose  $p \wedge q$  is a subformula of  $A$ , and consider how  $p$  and  $q$  are joined in  $B$ .

(1) It is impossible that  $p$  and  $q$  be disjunctively joined in  $B$ . Suppose they are. By Lemma 1, for some  $S_1, \dots, S_k$ , where  $S_i \in \{\top, \perp\}$ , we have that  $A_{S_1 \dots S_k}^{r_1 \dots r_k} \leftrightarrow (p \wedge q)$  is a tautology. On the other hand, by using Lemma 2 we infer that  $B_{S_1 \dots S_k}^{r_1 \dots r_k} \leftrightarrow C$  can be a tautology only if either  $C \leftrightarrow (p \vee q)$ , or  $C \leftrightarrow p$ , or  $C \leftrightarrow q$ , or  $C \leftrightarrow \top$ , or  $C \leftrightarrow \perp$ , is a tautology. So  $A_{S_1 \dots S_k}^{r_1 \dots r_k} \leftrightarrow B_{S_1 \dots S_k}^{r_1 \dots r_k}$  is not a tautology, which contradicts the assumption that  $A \leftrightarrow B$  is a tautology.

(2) It is also impossible that  $p$  and  $q$  be conjunctively joined in  $B$ , but not directly. Otherwise, we would have in  $B$  a subformula  $P \wedge Q$  or  $Q \wedge P$  with  $p$  in  $P$  and  $q$  in  $Q$ , and a subformula  $C \vee D$  of  $P$  with  $p$  either in  $C$  or in  $D$ ; we need not consider separately the analogous case when  $C \vee D$  is a subformula

of  $Q$  with  $q$  either in  $C$  or in  $D$ . Then  $A_{\perp}^p \leftrightarrow A'$  and  $B_{\perp}^p \leftrightarrow B'$  are tautologies for  $A'$  a formula of  $\mathcal{L}$  without  $q$  and  $b'$  a diversified formula of  $\mathcal{L}_{\wedge, \vee}$  with  $q$ . By Lemma 2,  $A' \leftrightarrow B'$  is not a tautology, which contradicts the assumption that  $A \leftrightarrow B$  is a tautology.

So  $p$  and  $q$  are directly conjunctively joined in  $B$ . So in  $\mathcal{S}_{\wedge, \vee}$  we have as a theorem  $B \leftrightarrow D$  for a diversified formula  $D$  of  $\mathcal{L}_{\wedge, \vee}$  with a subformula  $p \wedge q$ . Since  $A \leftrightarrow B$  is a tautology, it is clear that  $A \leftrightarrow D$  is a tautology too. If  $A_q^{p \wedge q}$  and  $D_q^{p \wedge q}$  are obtained from respectively  $A$  and  $D$  by replacing the single occurrence of  $p \wedge q$  by  $q$ , then  $A_q^{p \wedge q} \leftrightarrow D_q^{p \wedge q}$  is a tautology, because  $A_{\top}^p \leftrightarrow D_{\top}^p$  is a tautology. In  $A_q^{p \wedge q}$  we have  $k$  connectives, and so by the induction hypothesis we obtain that  $A_q^{p \wedge q} \leftrightarrow D_q^{p \wedge q}$  is a theorem of  $\mathcal{S}_{\wedge, \vee}$ . Hence  $(A_q^{p \wedge q})_{p \wedge q}^q \leftrightarrow (D_q^{p \wedge q})_{p \wedge q}^q$  is a theorem of  $\mathcal{S}_{\wedge, \vee}$ ; in other words,  $A \leftrightarrow D$  is a theorem of  $\mathcal{S}_{\wedge, \vee}$ , and since  $B \leftrightarrow D$  is such a theorem too, we obtain that  $A \leftrightarrow B$  is a theorem of  $\mathcal{S}_{\wedge, \vee}$ . We proceed analogously when  $p \vee q$  is a subformula of  $A$ .  $\neg$

A  $\neg \wedge \vee$ -equivalence is defined as a  $\wedge \vee$ -equivalence with  $\mathcal{L}_{\wedge \vee}$  replaced by  $\mathcal{L}_{\neg, \wedge, \vee}$ , and, as before, a  $\neg \wedge \vee$ -equivalence  $A \leftrightarrow B$  is *diversified* when  $A$  and  $B$  are diversified. The formal system  $\mathcal{S}_{\neg, \wedge, \vee}$  is defined as  $\mathcal{S}_{\wedge, \vee}$  save that the axioms and theorems are  $\neg \wedge \vee$ -equivalences instead of  $\wedge \vee$ -equivalences, and we have the additional axioms of the following forms:

$$\begin{aligned} \neg \neg A &\leftrightarrow A, \\ \neg(A \wedge B) &\leftrightarrow (\neg A \vee \neg B), & \neg(A \vee B) &\leftrightarrow (\neg A \wedge \neg B), \end{aligned}$$

and the following additional rule:

$$\frac{A \leftrightarrow B}{\neg A \leftrightarrow \neg B}$$

A formula of  $\mathcal{L}$  is called  $\neg$ -reduced when  $\neg$  occurs in it only before letters (i.e. only in subformulae of the form  $\neg p$ ). A letter occurs *positively* in a  $\neg$ -reduced formula when it is not in the scope of  $\neg$ ; otherwise it occurs *negatively*. We can prove the following.

**PROPOSITION 1**  $\neg$ . *A diversified  $\neg \wedge \vee$ -equivalence is a tautology iff it is a theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$ .*

**PROOF.** It is clear that every theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$  is a tautology. For the converse, we suppose that the diversified  $\neg \wedge \vee$ -equivalence  $A \leftrightarrow B$  is a tautology. It is easy to see that in  $\mathcal{S}_{\neg, \wedge, \vee}$  we have as theorems  $A \leftrightarrow A'$  and  $B \leftrightarrow B'$  for  $\neg$ -reduced formulae  $A'$  and  $B'$  of  $\mathcal{L}_{\neg, \wedge, \vee}$  (for that, the new axioms of  $\mathcal{S}_{\neg, \wedge, \vee}$  are essential), such that the diversified  $\neg \wedge \vee$ -equivalence  $A' \leftrightarrow B'$  is a tautology.

It is impossible that a letter  $p$  occurs positively in  $A'$  and negatively in  $B'$ . Suppose it does. Very much as in the proof of Lemma 1, we would obtain for some  $S_1, \dots, S_n$ , where  $S_i \in \{\top, \perp\}$ , that  $(A')_{S_1 \dots S_n}^{q_1 \dots q_n} \leftrightarrow p$  is a tautology. On the other hand,  $(B')_{S_1 \dots S_n}^{q_1 \dots q_n} \leftrightarrow C$  can be a tautology only if either  $C \leftrightarrow \neg p$ , or

$C \leftrightarrow \top$ , or  $C \leftrightarrow \perp$ , is a tautology. So  $(A')_{S_1 \dots S_n}^{q_1 \dots q_n} \leftrightarrow (B')_{S_1 \dots S_n}^{q_1 \dots q_n}$  is not a tautology, and hence  $A' \leftrightarrow B'$  is also not a tautology, contradicting what we have inferred above from the assumption that  $A \leftrightarrow B$  is a tautology. It is thereby impossible too that a letter occurs negatively in  $A'$  and positively in  $B'$ .

So the diversified  $\neg \wedge \vee$ -equivalence  $A' \leftrightarrow B'$  is an instance of a diversified  $\wedge \vee$ -equivalence  $A'' \leftrightarrow B''$ , which is a tautology. (Just replace every letter  $p$  occurring negatively in  $A'$  and  $B'$  by  $\neg p$ .) By Proposition 1, we have that  $A'' \leftrightarrow B''$  is a theorem of  $\mathcal{S}_{\wedge, \vee}$ . Hence  $A' \leftrightarrow B'$  is a theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$ , and since  $A \leftrightarrow A'$  and  $B \leftrightarrow B'$  are theorems of  $\mathcal{S}_{\neg, \wedge, \vee}$ , we obtain that  $A \leftrightarrow B$  is a theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$ .  $\dashv$

It is easy to see that the formal systems  $\mathcal{S}_{\wedge, \vee}$  and  $\mathcal{S}_{\neg, \wedge, \vee}$  are decidable formal systems. (For every formula of  $\mathcal{L}_{\neg, \wedge, \vee}$  we have a  $\neg$ -reduced normal form in  $\mathcal{L}_{\neg, \wedge, \vee}$  unique up to associativity and commutativity of  $\wedge$  and  $\vee$ .)

### 3 Isomorphic formulae with perfect generalizability

Let  $\mathcal{K}$  be a category whose objects are the formulae of  $\mathcal{L}_{\neg, \wedge, \vee}$ , and whose arrows  $f: A \rightarrow B$  are intuitively interpreted as deductions, or proofs, from  $A$  to  $B$ . We assume that  $\mathcal{K}$  has isomorphisms covering the theorems of  $\mathcal{S}_{\neg, \wedge, \vee}$ . This means, for example, that we have an isomorphism of the type  $(A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$ , whose inverse is of the type  $A \wedge (B \wedge C) \rightarrow (A \wedge B) \wedge C$ .

These isomorphisms of  $\mathcal{K}$  correspond to deductions in the multiplicative fragment of linear propositional logic. These are the basic arrows of  $\mathcal{K}$ , but  $\mathcal{K}$  can have more arrows than that. We assume however that  $\mathcal{K}$  does not go beyond deductions of classical propositional logic, which means that if  $f: A \rightarrow B$  is an arrow of  $\mathcal{K}$ , then  $A \rightarrow B$  is a tautology, with the arrow  $\rightarrow$  interpreted as the connective of material implication.

For a formula  $A$ , let  $|A|$  be the number of occurrences of letters in  $A$ . For every arrow  $f: A \rightarrow B$  of  $\mathcal{K}$ , if  $x$  is an occurrence of a letter in  $A$ , then let  $o(x) = n-1$  iff  $x$  is the  $n$ -th occurrence of letter in  $A$  counting from the left, and if  $x$  is an occurrence of a letter in  $B$ , then let  $o(x) = |A|+n-1$  iff  $x$  is the  $n$ -th occurrence of letter in  $B$  counting from the left.

We assume that every arrow  $f: A \rightarrow B$  of  $\mathcal{K}$  induces on the ordinal  $|A| + |B|$  an equivalence relation  $L_f$ , called the *linking relation* of  $f$ , which satisfies the condition that  $(o(x), o(y)) \in L_f$  *only if*  $x$  and  $y$  are occurrences in  $A$  or  $B$  of the same letter. A linking relation is *perfect* when instead of *only if* in this condition we have *if and only if*.

We say that  $A$  and  $B$  are *uniform instances* of  $A_1$  and  $B_1$  when they are instances of  $A_1$  and  $B_1$  respectively with the same letter-for-letter substitution (i.e. substitution that replaces a letter by a letter).

We say that an arrow  $f : A \rightarrow B$  is *generalized* to an arrow  $f_1 : A_1 \rightarrow B_1$  when  $A$  and  $B$  are uniform instances of  $A_1$  and  $B_1$ , and the linking relations  $L_f$  and  $L_{f_1}$  are the same.

We say that a category  $\mathcal{K}$  is *perfectly generalizable* when each of its arrows can be generalized to an arrow of  $\mathcal{K}$  with a perfect linking relation. Examples of perfectly generalizable categories with syntactically defined arrows will be given below after Lemma 3.

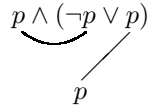
A category  $\mathcal{K}$  will be called *permutational* when for every isomorphism  $f : A \rightarrow B$  of  $\mathcal{K}$  the linking relation  $L_f$  corresponds to a bijection between  $|A|$  and  $|B|$ , and if  $g : B \rightarrow A$  is the inverse of  $f$ , then  $L_g$  corresponds to the inverse of this bijection. Permutational categories may be both perfectly generalizable and not perfectly generalizable. In this section we consider those of the first kind, while in the next one we deal with the second kind.

Permutational categories arise naturally whenever we have a certain modelling of categories that we are now going to describe. The category *SplPre* is the category whose arrows are *split preorders* between finite ordinals; *Gen* is a subcategory of *SplPre* whose arrows are *split equivalences* between finite ordinals, while *Rel* is a category whose arrows are relations between finite ordinals. (The category *Rel* has an isomorphic image within *SplPre* by a map that preserves composition, but not identity arrows.) A split preorder is a preordering relation on the disjoint union of two sets conceived as source and target, and analogously for split equivalences, which are equivalence relations. We have investigated *SplPre* and *Gen* systematically in [15], [11] and [12], and we have used *Gen* and *Rel* as model categories for equality of deductions in [13] and [14], and in other papers related to these two books.

We may use these model categories to produce the linking relation of  $\mathcal{K}$  in the following manner. For a functor  $G$  from  $\mathcal{K}$  into a model category such as *SplPre*, *Gen* or *Rel*, we take that  $(n, m) \in L_f$  iff the ordinals corresponding to  $n$  and  $m$  are linked by the reflexive, symmetric and transitive closure of  $Gf$ , which coincides with  $Gf$  if  $Gf$  is an equivalence relation of *Gen*. In this section we are interested in particular in linking relations produced by *Gen*, while in the next section we will encounter also one produced by *Rel*.

It may happen that the same category  $\mathcal{K}$  produces different linking relations with different functors  $G$ . The notions of perfectly generalizable and permutational category are relative to a chosen kind of linking relations.

Let us give an example of linking relations. We may have in  $\mathcal{K}$  an arrow  $f : p \wedge (\neg p \vee p) \rightarrow p$  corresponding to modus ponens, such that  $L_f$  will give the following linking





and another arrow  $g: p \wedge (\neg p \vee p) \rightarrow p$  corresponding to the first projection, such that  $L_f$  will give the following linking

$$\begin{array}{c} p \wedge (\neg p \vee p) \\ \searrow \\ p \end{array}$$

The arrow  $f$  can be generalized to an arrow of the type  $p \wedge (\neg p \vee q) \rightarrow q$ , while  $g$  can be generalized to an arrow of the type  $p \wedge (\neg q \vee r) \rightarrow p$ . Both of these arrows to which  $f$  and  $g$  are generalized have a perfect linking.

If the linking relations of the arrows of  $\mathcal{K}$  are produced by a functor  $G$  into  $SplPre$  (or a subcategory thereof), as explained above, then we may conclude that  $\mathcal{K}$  is permutational. This is because the isomorphisms of  $SplPre$  correspond to bijections. This is so both for perfectly generalizable and for not perfectly generalizable categories  $\mathcal{K}$ . We can prove the following lemma.

**LEMMA 3.** *If  $\mathcal{K}$  is a permutational perfectly generalizable category, and  $A$  and  $B$  are isomorphic in  $\mathcal{K}$ , then there are diversified formulae  $A_1$  and  $B_1$  such that  $A$  and  $B$  are uniform instances of  $A_1$  and  $B_1$  and  $A_1 \leftrightarrow B_1$  is a tautology.*

**PROOF.** Since  $\mathcal{K}$  is perfectly generalizable, the arrows  $f: A \rightarrow B$  and its inverse  $g: B \rightarrow A$  of  $\mathcal{K}$  are generalized to the arrows  $f_1: A_1 \rightarrow B_1$  and  $g_2: B_2 \rightarrow A_2$  with perfect linking relations. Since  $\mathcal{K}$  is permutational, these relations correspond to bijections inverse to each other. From that we conclude that  $A_1$ ,  $B_1$ ,  $A_2$  and  $B_2$  are diversified formulae. Since  $B$  is a letter-for-letter instance of both  $B_1$  and  $B_2$ , which are diversified,  $B_1$  is a letter-for-letter instance of  $B_2$ , and since the linking relations of  $f_1$  and  $g_2$  correspond to bijections inverse to each other, our letter-for-letter substitution produces  $A_1$  out of  $A_2$ . According to what we assumed at the beginning of the section,  $A_1 \rightarrow B_1$  and  $B_2 \rightarrow A_2$  are tautologies, and hence  $B_1 \rightarrow A_1$  is a tautology too.  $\dashv$

As categories  $\mathcal{K}$  covered by this lemma we have the free distributive lattice category of [13] (Section 11.1), which axiomatizes equality of deductions in conjunctive-disjunctive classical logic (the objects of this category are in  $\mathcal{L}_{\wedge, \vee}$ ), and the free proof-net category of [14] (Section 2.2), which axiomatizes equality of deductions in the multiplicative fragment of linear logic without propositional constants (its objects are in  $\mathcal{L}_{\neg, \wedge, \vee}$ ). One may easily conceive other such examples, and in particular the example of a category axiomatizing equality of deductions in the whole of classical propositional logic (whose objects are in  $\mathcal{L}_{\neg, \wedge, \vee}$ ). One would just take as equations of  $\mathcal{K}$  those equations  $f = g$  such that  $Gf = Gg$ , where  $G$  is a nontrivial functor from  $\mathcal{K}$  to  $Gen$  making  $\mathcal{K}$  perfectly generalizable.

The following proposition characterizes pairs of isomorphic formulae for permutational perfectly generalizable categories.

PROPOSITION 2 $\neg$ . *The formulae  $A$  and  $B$  are isomorphic in a permutational perfectly generalizable category iff  $A \leftrightarrow B$  is a theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$ .*

PROOF. From left to right, suppose that  $A$  is isomorphic to  $B$ . So, by Lemma 3, there are diversified formulae  $A_1$  and  $B_1$  such that  $A$  and  $B$  are uniform instances of  $A_1$  and  $B_1$ , and  $A_1 \leftrightarrow B_1$  is a tautology. By Proposition 1 $\neg$ , we obtain that  $A_1 \leftrightarrow B_1$  is a theorem of  $\mathcal{S}_{\neg, \wedge, \vee}$ , and hence  $A \leftrightarrow B$  is that too.

For the other direction, we have assumed that the equivalences of  $\mathcal{S}_{\neg, \wedge, \vee}$  are covered by the isomorphisms of our category.  $\dashv$

We have a Proposition 2 analogous to Proposition 2 $\neg$  with  $\mathcal{S}_{\neg, \wedge, \vee}$  replaced by  $\mathcal{S}_{\wedge, \vee}$ , for categories whose objects are the formulae of  $\mathcal{L}_{\wedge, \vee}$ . Since both  $\mathcal{S}_{\wedge, \vee}$  and  $\mathcal{S}_{\neg, \wedge, \vee}$  are easily seen to be decidable systems, Propositions 2 and 2 $\neg$  lead to a decision procedure for isomorphic formulae.

## 4 Isomorphic formulae without perfect generalizability

For  $f: A \rightarrow B$  an arrow, let the set of *diagonal links* of  $f$  be the set  $D_f = \{(n, n) \mid n \in |A| + |B|\}$ . We will here call an arrow  $f: A \rightarrow B$  in a category a *zero arrow* when its linking relation  $L_f$  is equal to the set  $D_f$  of its diagonal links. A *zero identity* arrow is an arrow  $0_A: A \rightarrow A$  with  $L_{0_A} = D_{0_A}$ . The *union* of the arrows  $f, g: A \rightarrow B$  will here be an arrow  $f \cup g: A \rightarrow B$  such that  $L_{f \cup g}$  is the transitive closure of  $L_f \cup L_g$ . (For a general treatment of zero arrows and union of arrows see [13].)

Consider a category  $\mathcal{K}$  such that the objects of  $\mathcal{K}$  are the formulae of  $\mathcal{L}$  and its arrows correspond to deductions in classical propositional logic. We assume that in  $\mathcal{K}$  we have for every  $A$  of  $\mathcal{L}$  a zero identity arrow  $0_A$ , and we have also closure under union of arrows. As a particular case of  $\mathcal{K}$  we have the category  $\mathbf{B}$  of [13] (Section 14.2), but possibly also a category that besides the zero arrows of the types  $\top \rightarrow \neg A \vee A$  and  $A \wedge \neg A \rightarrow \perp$ , like those we have in  $\mathbf{B}$ , has also arrows of this type with linking relations given by the following pictures:

$$\begin{array}{ccc} \top & & A \wedge \neg A \\ \frown & & \smile \\ \neg A \vee A & & \perp \end{array}$$

The linking relation  $L_f$  of an arrow  $f: A \rightarrow B$  of  $\mathbf{B}$ , which is an equivalence relation, is not the relation  $Gf \subseteq A \times B$ , where  $G$  is the functor into *Rel* with respect to which  $\mathbf{B}$  is coherent; i.e., the functor  $G$  such that for  $f, g: A \rightarrow B$  arrows of  $\mathbf{B}$  we have that  $f = g$  iff  $Gf = Gg$  (see [13], Section 14.2). We define  $L_f$  as the reflexive, symmetric and transitive closure of  $Gf$ . Note that for  $f, g: A \rightarrow B$  arrows of  $\mathbf{B}$  we do not have that  $L_f = L_g$  only if  $f = g$  in  $\mathbf{B}$ ,

but we still have that for particular arrows  $f$  and  $g$ , as in the proof of Lemma 7. What we need essentially is that this holds when one of  $f$  and  $g$  is an identity arrow.

For the categories  $\mathcal{K}$  other than  $\mathbf{B}$  we need the same assumption, which is guaranteed of course if equality of arrows in  $\mathcal{K}$  is defined via the linking relations—if we have namely that for  $f, g: A \rightarrow B$  the equation  $f = g$  holds in  $\mathcal{K}$  iff  $L_f = L_g$ . Our assumptions about  $\mathcal{K}$  must guarantee also that we have an analogue of Lemma 6 below. These assumptions can be quite standard, like those spelled out in [13] and [14]. If composition of these linking relations is defined naturally, like composition in the category *Gen* (see [15], Section 2, for a detailed study), then in  $\mathbf{B}$  we do not have that  $L_{g \circ f}$  is equal to the composition of  $L_f$  with  $L_g$ , but this may hold for other categories  $\mathcal{K}$ .

We take that our assumptions imply that  $\mathcal{K}$  is permutational in the sense of the preceding section. However,  $\mathcal{K}$  is not perfectly generalizable, because of the presence of zero identity arrows. The arrow  $0_p: p \rightarrow p$  cannot be generalized to one with a perfect linking relation. The category  $\mathbf{B}$  is permutational and is not perfectly generalizable.

The definition of the linking relation for the arrows of  $\mathbf{B}$  given above, and our notion of linking relation for  $\mathcal{K}$  in general, is motivated by the wish to have a uniform notion based, as in the preceding section, on an equivalence relation. This uniform notion is related to generality of deductions (more consistently in the preceding section than in the present one). If however we do not strive for this uniformity, then we may define  $L_f$  for  $\mathbf{B}$  just as  $Gf$ , and our proofs would still go through. For  $\mathcal{K}$  in general, we may also have a notion of linking relation like  $Gf$ .

We prove first a simple preliminary lemma.

LEMMA 4. *For every formula  $A$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  in which the letter  $p$  occurs there are formulae  $A_1$  and  $A_2$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  such that  $A \leftrightarrow ((p \wedge A_1) \vee A_2)$  is a tautology.*

PROOF. We proceed by induction on the number  $n$  of occurrences of binary connectives in  $A$ . If  $n = 0$ , then  $A$  is  $p$ , and we take  $A_1$  and  $A_2$  to be respectively  $\top$  and  $\perp$ .

Suppose  $n > 0$ . If  $A$  is  $A' \wedge A''$  with an occurrence of  $p$  in  $A'$ , then by the induction hypothesis we have formulae  $A'_1$  and  $A'_2$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  such that  $A' \leftrightarrow ((p \wedge A'_1) \vee A'_2)$  is a tautology. We take  $A_1$  and  $A_2$  to be respectively  $A'_1 \wedge A''$  and  $A'_2 \wedge A''$ . If  $A$  is  $A' \vee A''$  with an occurrence of  $p$  in  $A'$ , then by the induction hypothesis we have formulae  $A'_1$  and  $A'_2$  as before, and we take  $A_1$  and  $A_2$  to be respectively  $A'_1$  and  $A'_2 \vee A''$ . If  $A$  is  $A'' \wedge A'$  or  $A'' \vee A'$  with an occurrence of  $p$  in  $A'$ , then we have that  $A \leftrightarrow (A' \wedge A'')$  or  $A \leftrightarrow (A' \vee A'')$  is a tautology, and we proceed as before.  $\dashv$

The tautology of this lemma is provable in every logic that algebraically corresponds to distributive lattices with top and bottom. We have next the following

lemmata.

LEMMA 5. *If  $B'$  is obtained from a formula  $B$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  by replacing a single occurrence of  $q$  by  $p \wedge q$ , then  $(p \wedge B) \rightarrow B'$  is a tautology.*

PROOF. We proceed by induction on the number  $n$  of occurrences of binary connectives in  $B$ . If  $n = 0$ , then  $B$  is  $q$ , and  $(p \wedge q) \rightarrow (p \wedge q)$  is a tautology.

Suppose  $n > 0$ . If  $B$  is  $B_1 \wedge B_2$  with the replaced occurrence of  $q$  in  $B_1$ , then by the induction hypothesis we have that  $(p \wedge B_1) \rightarrow B'_1$  is a tautology, and hence, by relying on the associativity of  $\wedge$ , we have that  $(p \wedge (B_1 \wedge B_2)) \rightarrow B'$ , where  $B'$  is  $B'_1 \wedge B_2$ , is a tautology too. If  $B$  is  $B_1 \vee B_2$  with the replaced occurrence of  $q$  in  $B_1$ , by the induction hypothesis we have again that  $(p \wedge B_1) \rightarrow B'_1$  is a tautology, and since  $(p \wedge (B_1 \vee B_2)) \rightarrow ((p \wedge B_1) \vee B_2)$  (which corresponds to a dissociativity arrow; see [13], Chapter 7) is a tautology, we have that  $(p \wedge (B_1 \vee B_2)) \rightarrow B'$ , where  $B'$  is  $B'_1 \vee B_2$ , is a tautology too. If  $B$  is  $B_2 \wedge B_1$  or  $B_2 \vee B_1$  with the replaced occurrence of  $q$  in  $B_1$ , then we have that  $B \rightarrow (B_1 \wedge B_2)$  or  $B \rightarrow (B_1 \vee B_2)$  is a tautology, and we proceed as before.  $\dashv$

LEMMA 6. *If  $A$  and  $B$  are formulae of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$ , with  $x$  and  $y$  occurrences of the letter  $p$  in respectively  $A$  and  $B$ , and  $A \rightarrow B$  is a tautology, then there is an arrow  $f: A \rightarrow B$  of  $\mathbf{B}$  such that in  $L_f - D_f$  we find only the pairs  $(o(x), o(y))$  and  $(o(y), o(x))$ .*

PROOF. By Lemma 4, we have formulae  $A_1$  and  $A_2$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  such that  $A \leftrightarrow ((p \wedge A_1) \vee A_2)$  is a tautology. The proof of this lemma yields an arrow  $\tau: A \rightarrow (p \wedge A_1) \vee A_2$  such that  $(o(x), |A|) \in L_\tau$  (note that if  $z$  is the leftmost occurrence of  $p$  in  $(p \wedge A_1) \vee A_2$ , i.e. the occurrence written down, then  $o(z) = |A|$ ). We also have arrows  $\sigma: (p \wedge A_1) \vee A_2 \rightarrow A$  and  $g: A \rightarrow B$  of  $\mathbf{B}$ , because their types correspond to tautologies. If  $\iota_1: p \wedge A_1 \rightarrow (p \wedge A_1) \vee A_2$  and  $\iota_2: A_2 \rightarrow (p \wedge A_1) \vee A_2$  are injection arrows, then for

$$\zeta_i =_{df} 0_B \circ g \circ \sigma \circ \iota_i,$$

where  $i \in \{1, 2\}$ , we have  $L_{\zeta_i} = D_{\zeta_i}$ , because the composition ends with  $0_B$ . If  $\pi: p \wedge A_1 \rightarrow p$  is a first projection arrow, then for  $\langle \pi, \zeta_1 \rangle: p \wedge A_1 \rightarrow p \wedge B$  we have

$$L_{\langle \pi, \zeta_1 \rangle} - D_{\langle \pi, \zeta_1 \rangle} = \{(0, |A_1| + 1), (|A_1| + 1, 0)\}$$

(this means that the two occurrences of  $p$  that are written down are linked).

If  $B'$  is obtained from  $B$  by replacing the occurrence  $y$  of  $p$  in  $B$  by  $p \wedge p$ , then the proof of Lemma 5 yields an arrow  $\eta: p \wedge B \rightarrow B'$  of  $\mathbf{B}$ . Out of the first projection arrow  $\pi': p \wedge p \rightarrow p$  we obtain an arrow  $\theta: B' \rightarrow B$  such that for  $\theta \circ \eta: p \wedge B \rightarrow B$  we have  $(0, o(y)) \in L_{\theta \circ \eta}$  (i.e. the occurrence of  $p$  written down in the source  $p \wedge B$  and the occurrence  $y$  in the target  $B$  are linked).

For

$$\mu =_{df} [\theta \circ \eta \circ \langle \pi, \zeta_1 \rangle, \zeta_2]: (p \wedge A_1) \vee A_2 \rightarrow B$$

we have  $L_\mu - D_\mu = \{(0, o(y)), (o(y), 0)\}$ , and we take  $f: A \rightarrow B$  to be  $\mu \circ \tau$ . We have  $L_{\mu \circ \tau} - D_{\mu \circ \tau} = \{(o(x), o(y)), (o(y), o(x))\}$ .  $\dashv$

The formulae  $A$  and  $B$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  are *letter-homogeneous* when every letter that occurs in  $A$  occurs in  $B$  the same number of times, and vice versa, every letter that occurs in  $B$  occurs in  $A$  the same number of times. Two  $\neg$ -reduced formulae of  $\mathcal{L}$  are letter-homogeneous when they are uniform instances of two letter-homogeneous formulae of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$ . This means that such formulae share both positive and negative occurrences of letters. We can prove the following.

LEMMA 7. *For  $A$  and  $B$  formulae of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$ , if  $A \leftrightarrow B$  is a tautology and  $A$  and  $B$  are letter-homogeneous, then  $A$  and  $B$  are isomorphic in  $\mathbf{B}$ .*

PROOF. Suppose  $A \rightarrow B$  is a tautology and  $A$  and  $B$  are letter-homogeneous. For every bijection mapping the occurrences of letters in  $A$  to the occurrences of the same letters in  $B$ , there is an arrow  $f: A \rightarrow B$  of  $\mathbf{B}$  such that  $L_f$  corresponds to this bijection. This is guaranteed by Lemma 6 and the operation of union of arrows of  $\mathbf{B}$ . Since the same holds for the tautology  $B \rightarrow A$  and the inverse of our bijection, we conclude that there is an arrow  $g: B \rightarrow A$  of  $\mathbf{B}$  such that  $L_g$  corresponds to this inverse. With the help of Boolean Coherence (see [13], Section 14.2; this is the assertion that the functor  $G$ , on which the definition of  $L_f$  is based, is a faithful functor from  $\mathbf{B}$  to  $\mathbf{Rel}$ ), we infer from  $L_{g \circ f} = L_{1_A}$  and  $L_{f \circ g} = L_{1_B}$  that  $f$  and  $g$  are isomorphisms, inverse to each other.  $\dashv$

We can now prove the following.

PROPOSITION 3. *For  $A$  and  $B$   $\neg$ -reduced formulae of  $\mathcal{L}$  we have that  $A$  and  $B$  are isomorphic in  $\mathbf{B}$  iff  $A \leftrightarrow B$  is a tautology and  $A$  and  $B$  are letter-homogeneous.*

PROOF. The direction from left to right is an easy consequence of the fact that the arrows of  $\mathbf{B}$  correspond to implications that are tautologies, of the fact that no arrow of  $\mathbf{B}$  links a positive occurrence of a letter in the source with a negative occurrence of a letter in the target, and vice versa, and of the fact that  $\mathbf{B}$  is a permutational category in the sense of Section 3.

For the other direction we use Lemma 7 and appeal to the fact that if the right-hand side holds, then there are formulae  $A_1$  and  $B_1$  of  $\mathcal{L}_{\top, \perp, \wedge, \vee}$  such that  $A_1$  and  $B_1$  are letter-homogeneous,  $A$  and  $B$  are uniform instances of  $A_1$  and  $B_1$  and  $A_1 \leftrightarrow B_1$  is a tautology. We derive that  $A_1 \leftrightarrow B_1$  is a tautology by appropriate substitutions within  $A \leftrightarrow B$ .  $\dashv$

Every formula of  $\mathcal{L}$  may effectively be reduced to a  $\neg$ -reduced formula isomorphic in  $\mathbf{B}$ . So Proposition 3 gives a characterization of arbitrary pairs of isomorphic formulae of  $\mathbf{B}$ , and leads to a decision procedure for checking whether such an isomorphism exists.

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